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ting with symbols, teachers may prevent the work from becoming mechanical by requiring the pupil to state the reason for each step. This takes time, but it increases the pupil's power of expression, and leads him to see that algebra is a science, and not a mere juggling with symbols.

Right here, it seems to me, is the secret of successful work in algebra. Teach algebra as a science, and classes will take greater interest in the subject and give greater returns in ground covered. Emphasize such points as the number of roots to be expected from an equation; show that $a^0 = 1$ is not a mystery, but a most reasonable convention.

Teach devices, and urge the pupils to use methods that require thought rather than manual labor. Keep in mind the mottoes. "Take a good look at the problem," and "Don't multiply until you have to," using such problems as

$$\frac{(a+b)^2(x+1)+(a+b)(x+1)+(x+1)}{a+b+1}=(a+b)^2+(a+b)+1$$

$$\frac{x+3}{x+1}-\frac{x+4}{x+2}=\frac{x-5}{x-3}-\frac{x-6}{x-4}$$

$$18(x+1)^2(x+2)^2=8(x-3)^2(x+1)^2$$

Above all, let the teacher himself know and love his subject. In many classes the work drags because the teacher is merely serving time. Let him know more than he is called upon to teach, and let him give his pupils some idea of the part which algebra is to play in their future work.

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MODERN TENDENCIES IN THE TEACHING OF ALGEBRA.*

BY FLETCHER DURELL.

At present two movements are prominent, or at least noticeable, in the teaching of algebra: (1) The movement which would make the content or subject-matter less artificial and more practical, or which in other words would put algebra in closer accord with the practical sciences and the breadwinning trades (closely connected with this and perhaps part of it are the correlation and laboratory movements); (2) the movement which looks to a better logical development of algebra, a closer definition of its concepts, and a more rigorous treatment of its principles.

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* Read at the meeting of the New York Section.

But there is another aspect of the matter which is bound strongly to influence the movements just named, even if in time it does not come to underlie and include them. This may be called the utility or functional method of treating the teaching of algebra. This method puts the uses, offices or functions of algebraic concepts and processes in the first place, and the exactness of their relations, that is, proofs of all kinds, in the second. According to it, an axiom is not so much a fundamental equivalence or source of certainty as it is a fundamental economy or tool. Even when an axiom is a source of certainty, this certainty is but the means to new utility. The functional method is largely coincident with what is called the humanistic method which is attracting attention in education. The humanistic method is so called because it appeals not merely to narrow intellectualism on the one hand or to narrow industrialism on the other, but to the whole man. The functional or humanistic method puts the physiology of a subject before its anatomy, uses and functions before structures. It is closely related to the philosophy of pragmatism which is itself an outgrowth or consequence of the philosophy of evolution. It is influencing other departments of education, as may be noted in such text-books as Millikan and Gale's " Physics," Reed and Kellogg's " English Grammars," and Judd's " Psychology," and seems likely to have a profound influence on education as a whole.

We will now give a few illustrations with a view to showing what the functional method means when applied to the study of algebra. Our illustrations are selected mainly with a view to making clear our leading idea and not with a view to completeness or consecutive order.

Take, for instance, the case of the zero exponent. In the cremations of subjects of study which freshmen and sophomore classes used to hold twenty years ago, algebra was prominent, and prominent in the cremation of algebra was that of x^0.

Pupils used frequently to ask the question how x^0 could have any meaning; if the exponent denotes the number of times a factor is taken how can a factor be taken zero times? Let us look at this matter from the functional point of view. If x^3 be divided by x^3 from one point of view we get unity as the quotient, or if we subtract exponents we get x^0. It is merely a matter of convenience which form of the result we use. x^0 is not a metaphysical mystery; it is only a convenient and efficient way of writing the result of the division of some power of x by itself. It finds a use, for instance, when we are writing a polynomial expression, in showing that the absolute term may be taken as continuous with the x terms. In other words x^0 is not to be interpreted metaphysically but functionally and practically. After such an interpretation it is accepted as readily and used as freely by pupils as any other power of x.

Similarly a negative exponent is seen to be merely a means or instrument to enable us to write a fraction in a form often more convenient than the customary one; also a fractional exponent may be regarded at the outset as only a convenient way of combining a power and a root of the same number. After preliminary difficulties have been cleared away in this manner, and after pupils have got their general utility bearings, they are much more ready to study the so-called proofs, or what I would call methods of establishing equivalence relations among various kinds of exponents (and hence of setting up indirect routes of applications for their utilities), and also for various later generalizations of exponents, and for a closer logical examination of their nature.

Taking up another illustration—in studying elementary algebra many pupils have difficulty in learning to use a parenthesis and its contents as a single quantity, as for instance in the multiplication of $(2x+3a)+5b$ by $(2x+3a)-5b$. If we take the functional view of this matter and realize that the office of the symbol ( ) is to combine a number of objects into a whole or unity we realize that the symbol in the present case is not sufficiently expressive of unity to the pupil to accomplish its purpose. Hence we may vary the symbol so as to make it more expressive of its function, convert it into a closed oval and get for example $(\overline{\underline{2x+3a}})+5b$ by $(\overline{\underline{2x+3a}})-5b$. If this change still does not make the parenthesis sufficiently expressive, for it and its contents we may use p, the first letter of the word parenthesis, and write $p + 5b$, $p - 5b$, reversing the substitution after the multiplication.

Similarly the meaning and spirit (function) of other symbols may be grasped, and their use varied and extended. This not only helps in the teaching of algebra but enlarges the field of its application to life.

Let me give a simple illustration of this use of the broader spirit of algebra which may be inculcated. A certain business house had an address of, say, 17–31 Barclay St. They advertised extensively in magazines and wished to be able to compare the returns from the different magazines. They found it impossible to induce any considerable number of their correspondents to state which particular advertisement had attracted their attention. Hence the firm followed the plan of giving a slightly different address for their business in each advertisement. Thus in *Harper's Magazine* they advertised the address of the house as 17 Barclay St., in *McClure's Magazine* as 19 Barclay St., in *Scribner's* as 21, etc. Hence the replies could be grouped by the addresses used on them. This was in effect an automatic algebraic device for grouping objects.

In like manner in the more strictly cultural field this method if fully carried out leads to a realization of all language (except the fine arts in part) as essentially a form of algebra, and the study of algebra comes to throw a new and fundamental light on the study and use of language.

As another illustration let us now apply our principle of putting the offices or functions of objects in the first place to the teaching of imaginaries. We first clear away the old metaphysical difficulties that hang around the topic by asking the pupil to set aside the psychological-metaphysical word "imaginaries" and substitute for it a visual graphical term like "perpendicular numbers." Thus we explain in the usual way that if OA is $+1$ and OA' is -1, we may conceive OA as converted into OA' by a rotation through 180°. But we may also conceive this rotation as subdivided into two steps or two rotations of 90° each, one of which, if performed on the result of the first, converts OA into OA'. Hence, since $\sqrt{-1}$ means a rotation through 90°, when applied once to OA it gives OB in the perpendicular position. But from the broad functional point of view it means more than this: it means that instead of dealing with simple positive and negative numbers represented by a line running to the right and left, we have enlarged our domain of number to something infinitely greater—a plane surface. Hence in the use of imaginaries we have found a vast storehouse of auxiliary quantities to draw on. If the annexation of negative numbers to the positive numbers of arithmetic has already given us so much new power in early algebra, what ought not this infinitely larger magazine of auxiliary quantity to do for us? After pupils have thus got their broad utility bearings they are interested in the means used to develop the power latent in this storehouse of toolage. They readily accept the mechanical rule that as compared with radicals the only change in methods of operation is to make the product of the two minus signs under a radical sign a minus sign outside (for example $\sqrt{-2} \times \sqrt{-3} = -\sqrt{6}$); accept the use of i as an abbreviation, and the use of the cycle or group of powers of i, etc. The pupil by this time has become indifferent to the name by which these quantities are known. You may call a concept or tool what you please provided the pupil understands its functions or uses. Thus in the treatment of this topic the use of this idea of functionality tends to remove some difficulties, to arouse a broad and pervasive interest, to give large cultural values, and at the same time to lead up to technical applications for special classes of students.

As another illustration let us consider next a formula like the one for the sum of n terms of an A.P., $S = n/2(a + l)$. The direct function of this formula is to convert addition into multiplication, a longer process into a shorter. Even so obvious a utility as this seems seldom to be brought specifically to the pupil's attention, it being left to filter in as a result of mere impact. But we also have here an example of a deeper functionality and one whose importance can hardly be overestimated. Why is it that if we know the two formulas $l = a + (n - 1)d$, $S = n/2(a + l)$, we know all of A.P.? Why is it that the three cases of percentage can be combined into one by formula-equation $p = b \times r$, and the four cases of interest into another, viz: $p = i \cdot r \cdot t$. This result is obtained not merely because the symbols involved are small and the formula a kind of shorthand. An important part of the explanation lies in the fact that the symbols p, b, r are approximately uniform in size and that there is among them not only uniformity of size, but symmetry in their positional relations. Hence the eye not only grasps them all simultaneously but is ready to treat all alike, giving each the lead in turn, etc. A realization of this enables the pupil to understand that in each formula of science he not only has a shorthand expression for a multiplitude of phenomena, but that he also has many formulas in one, as many as there are letters in the given formula. It becomes plain in time also to him that these useful properties or functions give rise to a dialectic or aggressive onward development of other functions, as by the generalization of coefficients or exponents or number of terms, by the combination of more or less similar formulas, and by the re-use of their properties or of the properties of the phenomena which one represents in connection with another. When the many useful functions of formula-equations are thus realized, pupils not only have no dislike of them, but realize them as a useful help and are glad to state a group of related phenomena in the formula shape whenever they can.

Let us now apply the idea of functionality to the fundamental laws of algebra, called the associative, distributive and communative laws. Here again we may begin by clearing away all ideas of metaphysical necessity in these laws. This may be done by the use of the conventional illustrations or by purely functional illustrations such as the following which seem to appeal more strongly to the pupil. Thus we may bring out that so far as actual work is concerned the multiplication of 3 by 7,295 does not equal 7,295 multiplied by 3. That is, the multiplication of 7,295 by 3 is equivalent to the multiplication of 3 by 7,295 in some respects but not in others. Also in using a table (a traverse table, for instance) where the last figure is approximate sometimes $10 \times 2 \neq 2 \times 10$.

Thus it is brought home to the pupil that instead of laws these principles might perhaps better be named customary or conventional methods of arranging symbols. They are used because they save labor and give new power. Thus in transforming $3(a + b) - 2(a - b)$ into the successive form $3a + 3b - 2a + 2b$, $3a - 2a + 3b + 2b$, $a + 5b$ we are following methods of ordering and combining objects for which we are at liberty to substitute better ones if we can find them.

When grasped in this way it becomes clear that the laws of algebra contain implicit in them important first principles of order, arrangement and organization, and that the study of this part of algebra may have many useful applications to life at large.

Taking up a matter of somewhat different kind, it may be shown that the functional method reduces to system and control, and makes valuable in new ways processes which are usually treated in a more or less haphazard manner and regarded as a matter of common sense.

One of the illustrations of the practical value of geometry which we are accustomed to give is that of determining the height of a tall building or steeple or tree by comparing the length of its shadow with the length of the shadow of some other object whose height is known. Yet how many of us have had occasion to measure the height of any object thus, or have ever earned a dollar or a loaf of bread thus or ever expect to? Nevertheless, there is inherent in this illustration a principle of utility which is of far-reaching importance, viz.: the use of auxiliary quantity to achieve a result difficult or impossible. This principle of the use of adapted auxiliary quantity the physicist uses constantly; so does the chemist. It is omnipresent in all mechanic arts and in business; it is the basis of ethics and philosophy. In fact, we all need to use it every hour, often many times an hour. Anything that will enable us to improve our use of adapted auxiliary quantities or objects is of fundamental and comprehensive importance. Perhaps mathematics, by the relative simplicity of its concepts and by its freedom from entangling alliances is best fitted to give thorough control of this implement. Here possibly we glimpse a category of utility or practicality which underlies and includes both vocational and cultural values as special cases or details.

Similarly one of the practical applications of quadratic equations commonly given is that of determining the time it would take two pipes to fill a tank if some relation between the rates of the two pipes is given. Here again actual use of this problem in practical life is not likely, but in the process of completing the square by the aid of which the problem is solved we again have the use of auxiliary quantity but in a more definite and controllable form than in geometry.

The broad, practical and cultural as distinguished from merely technical values which result from such a study of algebra may be illustrated thus. A certain business man, *A*, had a creditor, *B*, of doubtful solvency, who owed him $100. After much effort *A* succeeded in inducing *B* to give him a check for the $100 drawn on a certain bank. *A* presented the check at the bank, but found that *B*'s balance there was only $90. *A*, who had studied the subject of completing the square in algebra, promptly deposited $10 to the credit of *B* and had his check cashed, saving $90 by the operation. Every person has need to use algebra or algebraic methods, more or less similar to this every day.

In algebra we find other illustrations of the definite and progressive ways in which we can learn to use auxiliary quantities; in certain cases of factoring, in transposition (a disguised form of the use of auxiliary quantity), in the addition and subtraction method of elimination, in rationalizing denominators, in the use of variables with respect to limits, in the proofs or formulas for A.P. and G.P., in logarithms, etc.

So much, then, for some simple illustrations of the aspect of the subject which we are trying to present. We may now say a word as to the relation of this method to the other tendencies in the teaching of algebra, noticeable at the present time, which were named at the outset.

The treatment of algebra in the way just described it is believed does not conflict with the movement which would make the content of algebra more directly practical. Rather it welcomes every algebraic problem taken from the mechanic arts as a concentrated illustration and justification of itself. Reciprocally, appreciation of functional values gives a broader meaning and value to such concrete applications. Also the functional treatment should welcome every new refinement in the logic of the subject, since every such refinement rightly used is the source of new power.

However in this combination and coördination of methods some conflicts may arise and some adustments be called for. Thus a particular example or topic may be useful functionally but not be practical in the usual narrow sense of that term. For instance, it may be shown from a functional point of view that the identity sign is not needed or desirable in the early study of algebra and this the logicist may not be willing to admit. However, this is a field which I do not desire to enter at this time.

In closing allow me to state the leading advantages in the functional or broad utility method of studying algebra, enlarging somewhat on those which have been mentioned incidentally in the preceding statement.

1. We have the removal of certain logical difficulties and extended power to deal with logical processes. Besides getting rid of difficulties the pupil is set free and gains power to leap from leading point to leading point and to fill in with exact equivalence relations later on.

Mr. H. V. Gummere, who is teacher of mathematics in Drexel Institute, Philadelphia, tells me that he has an evening class of the more mature mechanics from the Baldwin Locomotive Works and similar establishments. In one course of study given them, he has two hours a week for six months in which to teach the elements of analytical geometry and of differential and integral calculus, the aim being that after a man has taken the course he shall be able to continue his studies of the calculus alone and become able to read a stiff mathematical book on electricity. Mr. Gummere says that many of the men after taking the course are able to do this. The practical instinct in the men is so highly developed that they can quickly take in the broad outlines of the calculus in its functional aspect and go on and develop these themselves.

2. It is believed that the functional method increases the interest of pupils in vital ways and gives the subject something like a constant sense of worthwhileness. Often general utility considerations like those presented above seem to touch the average pupil at a more vital point and in a more stimulating way than direct concrete applications. The subject of interest needs a thorough working-out. Pupils are more interested in geometrical drawing than they are in going out on the campus and determining the height of a steeple or the distance of an inaccessible object; often more interested in working out puzzles than in either. The law seems to be that pupils are most interested in that form of activity which gives or promises to give largest returns for least effort. This may explain why general utility considerations appeal to them strongly.

3. It gives a vital and essential but not a costly concrete correlation of studies. This is a correlation of and through functions and utilities.

4. It is readily seen that all functions vary with circumstances, hence this treatment tears apart the narrow rigid processes of the demonstrational treatment, and leaves the pupil free and stimulates him to variations and innovations. This, rather than the demonstrational one, seems to be the route along which some of the greatest discoveries in mathematics have been made. Hence our method gives new value to the study of the history of a subject.

5. Finally the method under consideration helps to define the cultural values of education, and put these values in a more definite and controllable shape. This is a help which is very much needed at this time.

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MODERN TENDENCIES IN THE TEACHING OF ALGEBRA.*

BY N. J. LENNES.

One of the most obvious facts about mathematics in our secondary schools is a very general dissatisfaction which is expressed on all sides. There is an alarming number of failures, especially in the first year of the high school, which argues that the pupils do not find the subject suited to their tastes and capacities. Instructors in the colleges and universities rarely miss an opportunity for declaring that their students came poorly prepared. The programs of teachers' meetings and the tables of contents of pedagogical journals are teeming with titles which assume that something is wrong.

Is this general faultfinding simply the natural condition of normal progress, the sign of healthy life, or is there at present an abnormal divergence between our practices and our best principles? and if the latter, what are the causes of this divergence? The main facts of the general history of pedagogy

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* Read at the meeting of the New York Section.